

DEL PEZZO SURFACES OF DEGREE 6

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ABSTRACT. We give a correspondence which associates, to each Del Pezzo surface X of degree 6 over a field k of characteristic 0, a collection of data consisting of a Severi-Brauer variety/ k and a set of points defined over some extension of k .

The main results in this paper, and specifically Theorem 5.1, give a way to describe Del Pezzo surfaces of degree 6 over a field k of characteristic 0, via a correspondence with objects (Severi-Brauer varieties) which can be understood in a completely explicit way if k is sufficiently nice (e.g. k a number field).

1. PRELIMINARIES

In this paper, we will deal with varieties V over a field k of characteristic 0. If L/k is a field extension, then we write V_L for the base extension $V \times_{\text{Spec } k} \text{Spec } L$, and \bar{V} for $V_{\bar{k}}$.

A Del Pezzo surface over a number field k is a smooth rational surface X whose anticanonical sheaf ω_X^{-1} is ample. To each Del Pezzo surface X is associated a number $d = (\omega_X, \omega_X)$ (where $(,)$ denotes intersection number), called the degree of X .

The results we need about Del Pezzo surfaces are summarized in the following proposition. We refer the interested reader to [Man74] for proofs and more details.

Proposition 1.1. *Let V be a Del Pezzo surface of degree d over a field k .*

- (a) $1 \leq d \leq 9$.
- (b) $\text{Pic } \bar{V}$ is a free abelian group of rank $10 - d$.
- (c) If $V' \rightarrow V$ is a birational morphism and V' is a Del Pezzo surface, then V is a Del Pezzo surface.
- (d) Either \bar{V} is isomorphic to the blowup of \mathbb{P}_k^2 at $r = 9 - d$ points $\{x_1, \dots, x_r\}$ in general position, or $d = 8$ and $\bar{V} \cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$. Conversely, if $r \leq 6$, then any surface satisfying this condition is a Del Pezzo surface of degree $d = 9 - r$. (For a set of $r \leq 6$ points, “general position” means that no three are collinear and no six lie on a conic.)
- (e) Let C be an exceptional curve; that is, C is a curve on \bar{V} such that $(C, C) = -1$ and $C \cong \mathbb{P}_k^1$. Then if $r \leq 6$, the image of C in \mathbb{P}_k^2 under the isomorphism of (d) is either: one of the x_i , a line passing through two of the x_i , or a conic passing through five of the x_i . Conversely, each point, line, and conic in this list gives rise to exactly one exceptional curve C .

Proof: These are (respectively) Theorem 24.3(i), Lemma 24.3.1, Corollary 24.5.2, Theorem 24, and Theorem 26.2 of [Man74]. ■

The assumption that $r \leq 6$ was made only to simplify the statements of (d) and (e); we will not be concerned with Del Pezzo surfaces of degree 1 or 2 in this paper.

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2. SEVERI-BRAUER VARIETIES: THE BASIC CONSTRUCTION

If V is a Del Pezzo surface over a field k , it is clear from the definition above that the exceptional curves on \overline{V} are preserved by the action of $G_k := \text{Gal}(\overline{k}/k)$; this information can be very useful in investigating properties of these surfaces.

Now let D be a Del Pezzo surface of degree 6. Let E_1, E_2 , and E_3 be the exceptional curves corresponding to the blow-ups of the three points $x_1, x_2, x_3 \in \mathbb{P}_k^2$ as in Proposition 1.1(d). Let F_{12} be the exceptional curve corresponding to the line between x_1 and x_2 , and define F_{13} and F_{23} similarly. By Proposition 1.1(e), the set $\{E_1, E_2, E_3, F_{12}, F_{13}, F_{23}\}$ is precisely the set of exceptional curves on D .

We can now examine the possibilities for blowing down these curves to obtain other, possibly simpler surfaces.

Proposition 2.1. *Let D be a Del Pezzo surface of degree six over a field k of characteristic 0. There is a field L such that $[L : k] = 1$ or 2 and surfaces X and Y defined over L such that the triple (D_L, X, Y) satisfies the following conditions:*

- (i) *there is a morphism $\pi_X : D_L \rightarrow X$ which exhibits D_L as the blow-up of X at a G_L -stable set of three non-collinear points $\{P_1, P_2, P_3\} \in X(\overline{k})$*
- (ii) *there is a morphism $\pi_Y : D_L \rightarrow Y$ which exhibits D_L as the blow-up of Y at a G_L -stable set of three non-collinear points $\{Q_1, Q_2, Q_3\} \in Y(\overline{k})$*
- (iii) *$\{\pi_X^{-1}(P_i) : 1 \leq i \leq 3\} \cup \{\pi_Y^{-1}(Q_i) : 1 \leq i \leq 3\}$ is a full set of six exceptional curves on \overline{D} .*
- (iv) *X and Y are Severi-Brauer varieties of dimension 2.*

Proof: Let L be the minimal field such that the sets $\{E_1, E_2, E_3\}$ and $\{F_{12}, F_{13}, F_{23}\}$ are both G_L -stable. Any element of G_k either fixes both sets or switches them, so L is either equal to k or quadratic over k . Let X' and Y' be the varieties obtained from blowing down $\{E_1, E_2, E_3\}$ and $\{F_{12}, F_{13}, F_{23}\}$, respectively, over \overline{k} . Then they can naturally be descended to varieties X and Y defined over L (see [Wei56] for details on descent). Properties (i)-(iii) are immediate.

To see property (iv), note that X and Y are Del Pezzo surfaces by Proposition 1.1(c). Now note that $\text{rank Pic } D = 4$ by Proposition 1.1(b), and blowing up at a point increases the rank of the Picard group by 1, so $\text{rank Pic } \overline{X}$ must be 1. Then by Proposition 1.1(b) the degree of X is 9, which means $r = 0$, so X is a twist of \mathbb{P}^2 . The same holds for Y . ■

Now we can also turn Proposition 2.1 around:

Proposition 2.2. *Let X be a Severi-Brauer variety over a field L and let $\{P_1, P_2, P_3\}$ be a G_L -stable set of non-collinear points in $X(\overline{L})$. Then there exist S and Y defined over L such that the triple (S, X, Y) satisfies conditions (i)-(iv) of Proposition 2.*

Proof: To obtain S , simply blow up X over L at the given set of points. To obtain Y , note that the three exceptional curves on \overline{S} which are the inverse images of $\{P_1, P_2, P_3\}$ form a G_L -stable set (call it C_1), and since the full set C of exceptional curves is G_L -stable, the complement $C \setminus C_1$ is also G_L -stable and can be blown down over L to obtain Y . Conditions (i)-(iii) are all obvious from the construction. ■

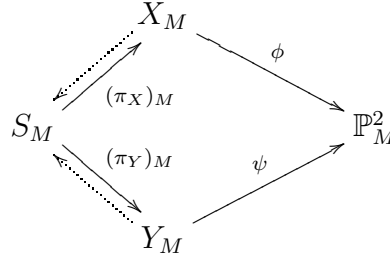
There is a natural one-to-one correspondence between (isomorphism classes of) Severi-Brauer varieties over L of dimension 2 and (isomorphism classes of) central simple algebras over L of dimension 9; both are parametrized by $H^1(G_L, \text{PGL}_3(\overline{L}))$. (Cf. [Ser79].)

Given the result of proposition 2.2, the natural question to ask is: how are the central simple algebras corresponding to X and Y related? The answer is our first main result.

Theorem 2.3. *Let X be a Severi-Brauer variety of dimension 2 over a field L , equipped with a G_L -stable set $\{P_1, P_2, P_3\}$ of three non-collinear points. Construct S and Y as in Proposition 2.1. Let x and y be the central simple algebras corresponding to X and Y respectively. Then $y = x^{\text{op}}$, the opposite algebra of x .*

Proof: We simply unravel the definition of the correspondence between Severi-Brauer varieties and central simple algebras. First, choose an ordering of the points $\{P_1, P_2, P_3\}$ and the points $\{Q_1, Q_2, Q_3\}$ of proposition 2 so that $\pi_X^{-1}(P_i) \cap \pi_Y^{-1}(Q_i) = \emptyset$ for all i .

Let M be the minimal Galois extension of L over which the P_i are each individually defined. Then the Q_i are all defined over M as well. Also, S_M is isomorphic to the blowup of \mathbb{P}_M^2 at the P_i , so we can choose a point $P \in S(M)$ which lies over a point in $\mathbb{P}^2(M) \setminus \{P_1, P_2, P_3\}$, so that P does not lie on any exceptional curve. Now $X_M \cong \mathbb{P}_M^2$, and since the automorphism group of \mathbb{P}_M^2 acts transitively on sets of four M -points in general position, we can construct an isomorphism $\phi : X_M \rightarrow \mathbb{P}_M^2$ sending the points $P_1, P_2, P_3, (\pi_X)_M(P)$ on X (notation as in Proposition 2.1) to $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$, and $(1 : 1 : 1)$, respectively. We can also construct an isomorphism $\psi : Y_M \rightarrow \mathbb{P}_M^2$ sending $Q_1, Q_2, Q_3, (\pi_Y)_M(P)$ on Y (notation as in Proposition 2.1) to $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$, and $(1 : 1 : 1)$, respectively.



Starting at \mathbb{P}_M^2 and going around counterclockwise in the diagram, we have a rational map $b := \psi \circ (\pi_Y)_M \circ (\pi_X)_M^{-1} \circ \phi^{-1}$. We can easily write down a formula for this map, as in [Har77], pp. 397-398:

$$b(x : y : z) = (yz : xz : xy).$$

So b is invariant under the natural action of $G_{M/L} := \text{Gal}(M/L)$.

But the composition $d := (\pi_Y)_M \circ (\pi_X)_M^{-1}$ is also $G_{M/L}$ -invariant. And $b = \psi \circ d \circ \phi^{-1}$, so

$$d = \psi^{-1} \circ b \circ \phi.$$

Now take $\sigma \in G_{M/L}$. Since ${}^\sigma d = d$ and ${}^\sigma b = b$, we have

$$\begin{aligned} \sigma(\psi^{-1} b \phi) &= \psi^{-1} b \phi \\ \psi({}^\sigma \psi^{-1}) &= b \phi({}^\sigma \phi^{-1}) b^{-1} \quad (1) \end{aligned}$$

Recall that the correspondence between Severi-Brauer varieties with points in M and central simple algebras split by M is via the cohomology group $H^1(G_{M/L}, PGL_3(M))$. The cocycles associated to X and Y are precisely $\eta_\sigma := \phi({}^\sigma \phi^{-1})$ and $\xi_\sigma := \psi({}^\sigma \psi^{-1})$. So (1) translates to:

$$\xi_\sigma = b \eta_\sigma b^{-1}. \quad (2)$$

Now, for any $\sigma \in G_{M/L}$, the set $\{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$ is stable under η_σ and ξ_σ , considered as automorphisms of \mathbb{P}^2 . So η_σ and ξ_σ land in the subgroup

$$H := \{A \in PGL_3(M) : \text{each row and column of } A \text{ has exactly one nonzero entry}\}.$$

(For simplicity, we often abuse notation and identify elements of $PGL_3(M)$ with representative matrices in $GL_3(M)$.)

Clearly H decomposes as a semi-direct product

$$H = H_D \rtimes H_P,$$

where H_D is the subgroup of diagonal matrices and H_P the subgroup of permutation matrices in $PGL_3(M)$. In particular, any matrix in H can be written $A = A_D A_P$, where A_D is diagonal and A_P is a permutation matrix.

Conjugation by b sends A_D to A_D^{-1} , and sends A_P to A_P . Note that

$$b A_D A_P b^{-1} = A_D^{-1} A_P = ({}^t A_D^{-1}) ({}^t A_P^{-1}),$$

or, in other words, conjugation by b is the same as applying the operator $A \mapsto {}^t A^{-1}$ on H .

Next we prove a lemma about this operator.

Lemma 2.4. *Let $c : G_{M/L} \rightarrow PGL_3(M)$ be a cocycle corresponding to a central simple algebra x . Precomposing c with the map $A \mapsto {}^t A^{-1}$ on $PGL_3(M)$ sends the class of x to the inverse of the class of x in $\text{Br}(L)$.*

Proof of lemma: By definition of the correspondence between c and x , c is constructed by the following formula: there is some isomorphism $\alpha : x \otimes_k \bar{k} \rightarrow M_3(\bar{k})$ such that for any $\sigma \in G_{M/L}$ and $A \in M_3(\bar{k})$,

$$c_\sigma A c_\sigma^{-1} = \alpha({}^\sigma \alpha^{-1})(A).$$

If we let $\beta(A) = {}^t \alpha(A)$, then β is an isomorphism $x^{\text{op}} \otimes \bar{k} \rightarrow M_3(\bar{k})$. And

$$\beta({}^\sigma \beta^{-1})(A) = {}^t((\alpha({}^\sigma \alpha^{-1}))({}^t A)) = {}^t(c_\sigma({}^t A) c_\sigma^{-1}) = {}^t c_\sigma^{-1} A ({}^t c_\sigma).$$

So the cocycle corresponding to x^{op} and β is precisely the cocycle obtained by precomposing c with the map $A \mapsto {}^t A^{-1}$. This proves the lemma. \square

Therefore, by the lemma applied to equation (2), η_σ and ξ_σ correspond to inverse classes in $\text{Br}(L)$, i.e. y is Brauer-equivalent to x^{op} . Since x and y are 9-dimensional, they are either both isomorphic to $M_3(L)$ or both division algebras. In the first case, $y \cong x^{\text{op}}$ trivially, and in the second case y is Brauer-equivalent to x^{op} , and two division algebras which are Brauer-equivalent are isomorphic. This proves the theorem. \blacksquare

From now on, we will denote by X^{op} the variety which corresponds to the central simple algebra opposite to the one corresponding to X ; X^{op} is unique up to isomorphism.

3. AUTOMORPHISMS OF SEVERI-BRAUER SURFACES

The next result we need is about the action of the automorphism group of a Severi-Brauer surface X on sets of three non-collinear points.

Theorem 3.1. *Let X be a 2-dimensional Severi-Brauer variety over a number field L , equipped with two G_L -stable sets $P = \{P_1, P_2, P_3\}$ and $Q = \{Q_1, Q_2, Q_3\}$ of non-collinear points. If $\xi : P \rightarrow Q$ is an isomorphism of L -varieties, ξ can be extended to an automorphism $\alpha \in \text{Aut}_L(X)$.*

Proof: Let M be the smallest Galois extension of L over which the points in P (and Q) are all individually defined. Let $G_{M/L} = \text{Gal}(M/L)$, as above. Since $X(M) \neq \emptyset$, we have an isomorphism $\phi : X_M \rightarrow \mathbb{P}_M^2$. As before, since $\text{Aut } \mathbb{P}_M^2$ acts transitively on sets of three points in general position, we may assume that P_1, P_2, P_3 go to whatever three non-collinear points we want. The following easy lemma provides those points:

Lemma 3.2. *Given a $G_{M/L}$ -set Z of order 3, we can find a set R of three non-collinear points in \mathbb{P}_M^2 such that R and Z are isomorphic as $G_{M/L}$ -sets.*

Proof of lemma: First note that we can immediately find a set of three distinct points $\{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbb{A}^1(M)$ which is invariant under $G_{M/L}$ and has the desired structure as a $G_{M/L}$ -set. Let $R_i = (1 : \alpha_i : \alpha_i^2)$. Then the R_i are non-collinear, and the $G_{M/L}$ -action on the R_i is the same as the action on the α_i , which is what we wanted. \square

Applying the lemma with $Z = P$, we obtain a set R of points with the same $G_{M/L}$ -action as the one on P . So set $\phi(P_i) = R_i$ for $i = 1, 2, 3$. We can also construct an isomorphism $\psi : X_M \rightarrow \mathbb{P}_M^2$ such that $\psi(Q_i) = R_i$ for $i = 1, 2, 3$. Make cocycles $\eta_\sigma = \phi(\sigma\phi^{-1})$ and $\xi_\sigma = \psi(\sigma\psi^{-1})$. We know that η_σ and ξ_σ are cohomologous in $H^1(G_{M/L}, \text{PGL}_3(M))$, since they both correspond to the same Severi-Brauer variety, and this cohomology group parameterizes Severi-Brauer varieties split by M . But in fact, from the construction of ϕ and ψ and the fact that P, Q , and R have the same $G_{M/L}$ -actions, we see that η_σ and ξ_σ can be viewed as cocycles in $Z^1(G_L, T)$, where

$$T = \{A \in \text{PGL}_3(M) : A(R_i) = R_i \text{ for } i = 1, 2, 3\}.$$

We will need to prove the following

Lemma 3.3. *The natural map $i : H^1(G_{M/L}, T) \rightarrow H^1(G_{M/L}, \text{PGL}_3(M))$ is injective.*

After the lemma is proved, we will conclude that η_σ and ξ_σ are cohomologous via a coboundary with image in T , i.e.

$$\xi_\sigma = B\eta_\sigma(\sigma B^{-1})$$

with $B \in T$, so that

$$\begin{aligned} \psi(\sigma\psi^{-1}) &= B\phi(\sigma\phi^{-1})(\sigma B^{-1}) \\ \sigma(\psi^{-1}B\phi) &= \psi^{-1}B\phi \end{aligned}$$

for all $\sigma \in G_L$. So $\psi^{-1}B\phi$ descends to an L -automorphism which extends φ .

Proof of lemma: First, let U be the set of matrices $B \in \text{GL}_3(M)$ such that the coordinate vectors in M^3 representing the R_i are eigenvectors of B . Then we get the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & M^* & \longrightarrow & U & \longrightarrow & T \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & M^* & \longrightarrow & \text{GL}_3(M) & \longrightarrow & \text{PGL}_3(M) \longrightarrow 1 \end{array}$$

Note that U is abelian (indeed, it is clearly conjugate to the subgroup of $\text{GL}_3(M)$ consisting of the invertible diagonal matrices). So we can pass to the long exact sequence of cohomology associated to this short exact sequence, part of which is

$$\begin{array}{ccccc}
H^1(G_{M/L}, U) & \longrightarrow & H^1(G_{M/L}, T) & \longrightarrow & H^2(G_{M/L}, M^*) \\
& & \downarrow i & & \parallel \\
& & H^1(G_{M/L}, PGL_3(M)) & \longrightarrow & H^2(G_{M/L}, M^*)
\end{array}$$

So if we can show that $H^1(G_{M/L}, U) = 0$, we'll have that the map at the top of the square is injective, which will imply that i is injective.

First note that $U = CK_D C^{-1}$, where K_D is the subgroup of diagonal matrices of $GL_3(M)$, and C is a change-of-basis matrix. So we have a group isomorphism $U \rightarrow M^* \times M^* \times M^*$ sending $CAC^{-1} \rightarrow (A_{11}, A_{22}, A_{33})$.

Let $R = \text{Spec } E$ as an L -variety. E is a three-dimensional L -algebra, and there are three distinct maps $M \otimes_L E \rightarrow M$ corresponding to the three elements of $R(M)$. Then the group homomorphism $M \otimes_L E \rightarrow M \times M \times M$ made from these three maps is an isomorphism of rings. Passing to the unit groups of both rings gives a group isomorphism $(M \otimes_L E)^* \rightarrow M^* \times M^* \times M^*$. It is easy to check that the composition $U \rightarrow M^* \times M^* \times M^* \rightarrow (M \otimes_L E)^*$ actually commutes with the action of $G_{M/L}$ on both sides.

Indeed, another way to see this composition is as the realization of U as the automorphism group of the line bundle over R_M corresponding to the invertible sheaf $\mathcal{O}_{R_M}(1)$. This automorphism group is isomorphic to $\mathcal{O}_{R_M}(R_M)^* = (M \otimes_L E)^*$.

But $H^1(G_{M/L}, (M \otimes_L E)^*) = 0$ by an extension of Hilbert's Theorem 90 (see [Ser79], X.1, ex. 2). This proves the lemma. \square \blacksquare

4. REVERSING THE CONSTRUCTION

Now we prove a result about recovering the Del Pezzo surface D from a suitably chosen Severi-Brauer variety X .

Theorem 4.1. *Let k be a field and let L/k be a quadratic extension with $\text{Gal}(L/k)$ generated by σ . Suppose X is a Severi-Brauer variety over L such that X and ${}^\sigma X$ correspond to opposite central simple algebras, and suppose we are given a G_L -stable set of non-collinear points $P := \{P_1, P_2, P_3\} \subseteq X(\bar{k})$. Then:*

(i) *The variety S we constructed in Proposition 2.2 can be descended to a Del Pezzo surface of degree 6 over k .*

(ii) *If we relax the requirements on the above set of data to let L be an étale algebra of degree 2 over k , then every Del Pezzo surface of degree 6 over k can be constructed in this way.*

Proof of theorem: In Proposition 2.2 we can take $Y = {}^\sigma X$, and by Theorem 3.1 we can assume that the set Q of blown-up points on Y is actually ${}^\sigma P$. So we have blowing-down maps $S \rightarrow X$ and $S \rightarrow {}^\sigma X$ as in the proposition, hence a map $\varphi : S \rightarrow X \times {}^\sigma X$. Now we prove

Lemma 4.2. *φ is a closed immersion.*

Proof of lemma: It is equivalent to show that $\varphi_{\bar{k}} : S_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$ is a closed immersion. So $\varphi_{\bar{k}}$ is the map which takes \mathbb{P}^2 blown up at three points and blows down each skew triple of exceptional curves in turn. This description of the map makes it clear that it is injective as a map of sets, and since blowups of projective schemes are projective, $\varphi_{\bar{k}}$ is projective; so φ

is projective and thus φ is a homeomorphism onto its image, a closed subset of $\mathbb{P}^2 \times \mathbb{P}^2$. We now need to check that the map on structure sheaves is surjective, which can be checked on the stalks.

What we need to check is that the map $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2, (\pi_X(P), \pi_Y(P))} \rightarrow \mathcal{O}_{S,P}$ induced by φ is surjective for all $P \in S(\overline{k})$. (For convenience, we assume for the remainder of the lemma that everything is over \overline{k} and drop subscripts.) If P lies on at most one of the exceptional lines, then one of the projections $p_S : S \rightarrow \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ restricts to an isomorphism of an open subset of S containing P (namely, S minus a skew triple of exceptional lines not containing P) onto its image. Thus

$$\mathcal{O}_{\mathbb{P}^2, p_S(P)} \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2, \varphi(P)} \rightarrow \mathcal{O}_{S,P}$$

is surjective, and so the latter map must be as well.

Now suppose P is one of the six points which lies on two exceptional lines. As in [Har77], p. 152, it is enough to check that the map $m_{\mathbb{P}^2 \times \mathbb{P}^2, \varphi(P)} \rightarrow m_{S,P}/m_{S,P}^2$ is surjective. Around P , S just looks like the blowup of \mathbb{A}^2 at a point, and so $m_{S,P}/m_{S,P}^2$ is two-dimensional, with generators which cut out the two exceptional lines going through P . Each of these two generators comes from exactly one of the maps $m_{\mathbb{P}^2, p_S(P)} \rightarrow m_{S,P}/m_{S,P}^2$ (whichever one does not collapse the line that that generator cuts out). So this implies the surjectivity of the map we want. \square

Now since φ is a closed immersion, it gives an isomorphism of S onto its image, which must be the graph of the birational map $b_1 : X \dashrightarrow S \rightarrow {}^\sigma X$ that blows up the P_i and then blows down the “other” three lines into the Q_i . (The graph of a birational map b is the *closure* of the set of points $\{(a, b(a)) | a \in \text{domain}(b)\}$.)

The same analysis shows that ${}^\sigma S$ is isomorphic to the graph of the birational map $b_2 : {}^\sigma X \dashrightarrow S \rightarrow X$ blowing up the Q_i and then blowing down the “other” three lines into the P_i . Since b_1 and b_2 are inverses by construction, the identification of $X \times {}^\sigma X$ with ${}^\sigma X \times X$ by changing the order of the factors induces a map f_σ from the graph of b_1 to the graph of b_2 . Then we obtain the following commutative diagram:

$$\begin{array}{ccc} S & \longrightarrow & X \times {}^\sigma X \\ \downarrow f_\sigma & & \downarrow \\ {}^\sigma S & \longrightarrow & {}^\sigma X \times X \\ \downarrow {}^\sigma f_\sigma & & \downarrow \\ S & \longrightarrow & X \times {}^\sigma X \end{array}$$

where the maps on the right are the isomorphisms arising from switching the factors. This shows that ${}^\sigma f_\sigma \circ f_\sigma$ is the identity, since the composition of maps on the right side is the identity. Therefore f_σ gives descent data for S , so that it can be descended (again, as in [Wei56]) to a k -variety. This proves statement (i).

As for statement (ii), this merely summarizes what we already know. Given a Del Pezzo surface D , if the field given in Proposition 2.1 was quadratic, we can take it to be L , and if that field was k , we can take $L = k \times k$. Then the Severi-Brauer varieties X and Y associated with D in Proposition 2.1 satisfy $X^{\text{op}} = Y = {}^\sigma X$ in either case. \blacksquare

5. OTHER RESULTS AND CONCLUSIONS

One remark that should be made about Theorem 4.1 is that the condition $\sigma X = X^{\text{op}}$ is not a very restrictive one. When $L = k \times k$, the varieties X satisfying the condition are generated by starting with any Severi-Brauer surface X'/k and then letting X be the disjoint union of X' and $(X')^{\text{op}}$.

When L is a quadratic field extension of k , we can view the Galois group generator σ as a linear automorphism of $\text{Br } L$. Suppose x is a class of order 3 in $\text{Br } L$. Then x can be written as $2(x + {}^\sigma x) + 2(x - {}^\sigma x)$, so

$$(\text{Br } L)[3] = (\text{Br } L)^{G_{L/k}}[3] \oplus W,$$

where W is the set of classes of algebras corresponding to varieties X satisfying the condition $\sigma X = X^{\text{op}}$.

In fact, a little more can be said: the spectral sequence

$$E_2^{p,q} := H^p(G_{L/k}, H^q(G_L, \overline{L}^*)) \Rightarrow H^{p+q}(G_k, \overline{k}^*)$$

yields the usual exact sequence

$$0 \rightarrow E_2^{2,0} \rightarrow \text{Br } k \rightarrow E_2^{0,2} \rightarrow E_2^{3,0},$$

but $E_2^{0,2} = (\text{Br } L)^{G_{L/k}}$ and $E_2^{3,0} = E_2^{1,0} = 0$ by Hilbert's Theorem 90 and the fact that L/k is cyclic. Since multiplication-by-3 is the identity on the 2-torsion group $E_2^{2,0}$, the natural map

$$(\text{Br } k)[3] \rightarrow (\text{Br } L)^{G_{L/k}}[3]$$

is an isomorphism, so that $(\text{Br } L)[3] \cong (\text{Br } k)[3] \oplus W$.

Finally, we simultaneously sum up the results we have established and include the proof of one last remark:

Theorem 5.1. *Giving a Del Pezzo surface of degree 6 over a field k of characteristic zero is equivalent to giving the following data:*

- (1) *an étale algebra L of degree 2 over k*
- (2) *a Severi-Brauer variety X of dimension 2 over L such that $\sigma X = X^{\text{op}}$, where σ generates $\text{Gal}(L/k)$*
- (3) *a subscheme P of X consisting of three geometric non-collinear points*

Moreover, two Del Pezzo surfaces S_i corresponding to L_i , X_i , and P_i ($i = 1, 2$) are isomorphic if and only if $L_1 \cong L_2$ and there is an isomorphism $X_1 \rightarrow X_2$ such that P_1 maps isomorphically onto P_2 .

Proof of theorem: All we need to check is the last statement. For the “if” direction, this simply follows from the description of S_i given in the proof of Theorem 4.1, as the graph of a birational map constructed in terms of X_i , ${}^\sigma X_i$, and P_i . For the “only if” direction, note that we constructed the objects L_i , X_i , and P_i intrinsically from S_i . If S_1 and S_2 are isomorphic, we naturally get the isomorphisms given in the statement of the theorem. (The only choice we made was between X and ${}^\sigma X$, but these varieties are isomorphic over L , and, as noted before, Theorem 3.1 implies that we can make the isomorphism send P to ${}^\sigma P$.) ■

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